A proof of the abc conjecture after Shinichi Mochizuki

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February 2023

Abstract

We give a survey of S. Mochizuki's ingenious inter-universal Teichmuller theory and ex- plain how it gives rise to Diophantine inequalities. The exposition was designed to be as self-contained as possible.(a complet part of (algebraic number theory papers part1- part 2)

1 Introduction

The author once heard the following observation, which was attributed to Grothen- dieck: There are two ways to crack a nut — one is to crack the nut in a single stroke by using a nutcracker; the other is to soak it in water for an extended period of time until its shell dissolves naturally. Grothendieck's mathematics may be regarded as an example of the latter approach.

Theorem 1. *(Vojta's Conjecture [Voj] for Curves, [IUTchIV, Corollary 2.3])*
Let X be a proper, smooth, geometrically connected curve over a number field; $D \subset X$ *a reduced divisor;* $U_X := X \setminus D$. *Write* ω_X *for the canonical sheaf on X. Suppose that* U_X *is a hyperbolic curve, i.e.,* $deg(\omega_X(D)) > 0$. *Then for any* $d \in \mathbb{Z}_{>0}$ *and* $\epsilon \in \mathbb{R}_{>0}$ *we have :*

$$ht_{\omega_{X(D)}} \lesssim (1+\epsilon)(log - diff_X + log - cond_D) \qquad (1)$$

Corollary 1. *(0.2.)(The abc Conjecture of Masser and Oesterl e [Mass1], [Oes])*
for any $\epsilon \in \mathbb{R}_{>0}$*we have*

$$max(|a|\,,|b|\,,|c|) \leq (\prod_{p|abc} p)^{1+\epsilon} \tag{2}$$

Proof. We apply Theorem 0.1 in the case where , $X = \mathbb{P}^1_{\mathbb{Q}} \supset D = \{0, 1, \infty\}$,and d=1 . We have $\omega_{\mathbb{P}^1}(D) = \mathcal{O}_{\mathbb{P}^{\not{k}}}(1), log{-}diff_{\mathbb{P}^1}(-a/b) = 0, log{-}cond_{\{0,1,\infty\}}(-a/b) = \sum_{p|a,b,a+b}(log(p))$ and $ht_{\mathcal{O}_{\mathbb{P}^1}}(-a/b) = log(max(|a|\,,|b|)) \simeq log(max(|a|\,,|b|),|c|)$ for a,b $\in \mathbb{Z}, b \neq 0, \because\ |a+b| \leq 2max(|a|, |b|)$. for $\epsilon > 0$, we take $\epsilon > \epsilon' > 0$. according to theorem 1. there exist a C , such that : $log(max(|a|\,,|b|\,,|c|)) \leq (1+\epsilon')\sum_{p|abc} log(p) + C$
$(log(max(|a|\,,|b|\,,|c|)) \leq \frac{1+\epsilon}{\epsilon-\epsilon'})$
, with a+b=c. . And this gives us the corollary. □

2 2-references

[IUTchI] S. Mochizuki, Inter-universal Teichmuller Theory I: Construction of Hodge The- aters. RIMS Preprint 1756 (August 2012). the latest version is available in http://www.kurims.kyoto-u.ac.jp/motizuki/papers-english.html
[IUTchII] S. Mochizuki, Inter-universal Teichmuller Theory II: Hodge-Arakelov-theoretic Evaluation. RIMS Preprint 1757 (August 2012). the latest version is available in http://www.kurims.kyoto-u.ac.jp/motizuki/papers-english.html
[IUTchIII] S. Mochizuki, Inter-universal Teichmuller Theory III: Canonical Splittings of the Log-theta-lattice. RIMS Preprint 1758 (August 2012). the latest version is available in http://www.kurims.kyoto-u.ac.jp/motizuki/papers-

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